Abstract. In many applied problems (geophysics, medicine, astronomy, etc) we cannot directly measure the values $x(t)$ of the desired physical quantity $x$ in different moments of time, so we measure some related quantity $y(t)$, and then we try to reconstruct the desired values $x(t)$. This problem is often ill-posed in the sense that two essentially different functions $x(t)$ are consistent with the same measurement results. So, in order to get a reasonable reconstruction, we must have some additional prior information about the desired function $x(t)$. Methods that use this information to choose $x(t)$ from the set of all possible solutions are called regularization methods.

In some cases, we know the statistical characteristics both of $x(t)$ and of the measurement errors, so we can apply statistical filtering methods (well-developed since the invention of a Wiener filter). In some situations, we know the properties of the desired process, e.g., we know that the derivative of $x(t)$ is limited by some number $\Delta$, etc. In this case, we can apply standard regularization techniques (e.g., Tikhonov’s regularization).

In many cases, however, we have only uncertain knowledge about the values of $x(t)$, about the rate with which the values of $x(t)$ can change, and about the measurement errors. In these cases, usually one of the existing regularization methods is applied. There exist several heuristics that choose such a method. The problem with these heuristics is that they often lead to choosing different methods, and these methods lead to different functions $x(t)$. Therefore, the results $x(t)$ of applying these heuristic methods are often unreliable.

We show that if we use fuzzy logic to describe this uncertainty, then we automatically arrive at a unique regularization method, whose parameters are uniquely determined by the experts knowledge. Although we start with the fuzzy description, but the resulting regularization turns out to be quite crisp.

1. INTRODUCTION

What is an inverse problem [TA77], [I83], [G84], [I86], [I86a], [LRS86], [CB86]. In many applied problems (geophysics, medicine, astronomy, etc) we cannot directly measure the values $x(t)$ of the desired physical quantity $x$ in different moments of time, so we measure some related quantity $y(t)$, and then we try to reconstruct the desired values $x(t)$. For example, in case the dependency between $x(t)$ and $y(t)$ is linear, we arrive at a problem of reconstructing $x(t)$ from the equation $y(t) = \int k(t, s)x(s) ds + n(t)$, where $k(t, s)$ is an approximately known function, and $n(t)$ denote the (unknown) errors of measuring $y(t)$. These problems are called inverse problems.

Another example of inverse problems is image reconstruction from a noisy raw data.

Why inverse problems are so difficult to solve? These problems are often ill-posed in the sense that two essentially different functions $x(t)$ are consistent with the same observations $y(t)$. For example, since all the measurement devices are inertial and thus suppress the high frequencies, the functions $x(t)$ and $x(t) + \sin(\omega t)$, where $\omega$ is sufficiently big, lead to almost similar values of $y(t)$. So, in order to get meaningful results, we must somehow choose from all possible solutions
methods proposed by A. N. Tikhonov and others [TA77], [I83], [G84], [I86], [I86a], [LRS86].

Inverse problems are extremely important for space exploration. If we are analyzing familiar processes, then we usually know (more or less) how the function $x(t)$ looks like. For example, we can know that $x(t)$ is a linear function $x(t) = C_1 + C_2 t$, or a sine function $x(t) = C_1 \sin(C_2 t + C_3)$, etc. In mathematical terms, we know that $x(t) = f(t, C_1, \ldots, C_k)$, where $f$ is a known expression, and the only problem is to determine the coefficients $C_i$. This is how, for example, the orbits of planets, satellites, comets, etc., are computed: the general shape of an orbit is known from Newton’s theory, so we only have to estimate the parameters of a specific orbit. In such cases, the existence of several other functions $x(t)$ that are consistent with the same observations, is not a big problem, because we choose only the functions $x(t)$ that are expressed by the formula $f(t, C_1, \ldots, C_k)$.

In space exploration one of the main objectives (and the main challenges) is to analyze new phenomena, new effects, qualitatively new processes, and in these cases no prior expression $f$ is known.

How these problems are traditionally solved? If we know the statistical characteristics of $x(t)$ and statistical characteristics of the measurement errors $n(t)$, then we can formulate the problem of choosing the maximally probable $x(t)$ and end up with one of the methods of statistical regularization, or filtering (Wiener filter is one of the examples of this approach).

If we do not have this statistical information, but we know, e.g., that the average rate of change of $x(t)$ is smaller than some constant $\Delta$ (i.e., $\sqrt{\int \dot{x}(t)^2 dt} \leq \Delta$), then we can apply regularization methods proposed by A. N. Tikhonov and others [TA77], [G84], [LRS86].

In particular, one of the most widely used (and most efficient) regularization techniques consists of choosing among all the $x(t)$ that are consistent with given observations, a function $x(t)$ for which the so-called Tikhonov functional (or Tikhonov stabilizer)

$$J(\mu) = a_0 \int (x(t))^2 dt + a_1 \int (\dot{x}(t))^2 dt + a_2 \int (x^{(2)}(t))^2 dt + \ldots + a_k \int (x^{(k)})^2 dt$$

takes the smallest possible value, where $a_i$ are non-negative real numbers, $a_k > 0$, $k \geq 1$, and $x^{(i)}(t)$ denotes $i$–th derivative of $x(t)$.

For image reconstruction problems, when instead of a function $x(t)$ of one variable $t$ we have a function $I(x, y)$ of two coordinates (that expresses brightness in a point $(x, y)$), a similar functional that involves partial derivatives can be used.

If no such information is available, it is usually recommended to use Tikhonov's (or alternative) regularization techniques that correspond to some values of $a_i$. Several semi-heuristic rules of choosing these parameters $a_i$ are known. The problem with these choices is that different rules sometimes lead to drastically different results, and therefore these results are unreliable.

Usually experts possess some uncertain knowledge. The whole situation seems hopeless, but it is not. Yes, in new fields we do not have precise knowledge of what is going on, but we may be able to make some uncertain predictions. For example, if we want to know how the temperature on a planet changes with time $t$, then the experts can tell that most likely, $x(t)$ is limited by some value $M$, and that the rate $\dot{x}(t)$ with which the temperature changes, is typically (or “most likely,”, etc) limited by some value $\Delta$, etc. We can also have some expert knowledge about the error, with which we measure $y(t)$, so the resulting expert’s knowledge about the value of $y(t)$ in some point $t$
looks like “the difference between the measured value $y(t)$ and the actual value $Y(t)$ is most likely, not bigger than $\delta$” (where $\delta$ is a positive real number given by an expert).

The importance of this information is stressed in [B92].

**What we are planning to do.** In the present paper we show that if we use fuzzy logic to describe this uncertainty, then we automatically arrive at a unique regularization method, whose parameters are uniquely determined by the experts knowledge. Moreover, although we start with the fuzzy description, but the resulting regularization turns out to be quite crisp.

In Section 2 we will discuss briefly how to choose an appropriate representation of the experts uncertainty, in Section 3 we use the resulting representations to solve the inverse problems.

## 2. PRELIMINARY DISCUSSION: HOW TO DESCRIBE RELATED UNCERTAINTY

**What we have to describe.** We want to use fuzzy logic to describe this kind of uncertainty. So we must do the following:

- find appropriate fuzzy representations of the experts statements of the type “most likely, $X$ is $\leq M$”, or “most likely, $|X - a| \leq \delta$”, where $X$ is unknown, and $M, a, \delta$ are known values;
- choose a way to combine the resulting fuzzy statements into a membership function for different $x(t)$;
- transform this fuzzy description of $x(t)$ into a single function $x(t)$ that will be produced as a solution of the inverse problem, i.e., choose an appropriate defuzzification.

In the present Section we will describe how to make all three choices. Actually we will start with choosing an appropriate combination rule, then we will choose an appropriate membership function, and then it will turn out that defuzzification is trivial.

**How to choose an aggregation function.** In general, our uncertain knowledge about the unknown function $x(t)$ consists of the statements of the following types: “most likely, $|x(t)| \leq M$”, “most likely, $|\dot{x}(t)| \leq \Delta$”, “most likely, $|y(t) - \int k(t, s)x(s)\,ds| \leq \delta$”, etc. Each statement is fuzzy in the sense that for an arbitrary function $x(t)$ we are not 100% sure whether this statement is true for this function or not. The general idea of fuzzy logic is to describe this uncertainty by a *membership function*, i.e., by a mapping that assigns to every $x(t)$ a number from the interval $[0,1]$, that describes to what extent we believe that this statement is true.

Suppose that we have already decided how to express each of previous statements in terms of membership functions. So we get a different membership function for each moment of time $t$ and for each statement. We must now generate a membership function that describes all our knowledge, i.e., that describes the fact that the first statement is true, and the second statement is true, etc. The total knowledge is obtained by applying “and” to all the statements, and therefore the resulting membership function must be obtained by applying one of the operations $\& : [0,1] \times [0,1] \rightarrow [0,1]$ that express “and” to all the correspondent membership functions $\mu_i(t): \mu(x(t)) = \mu_1(x(t))\&\mu_2(x(t))\&...$

Experimental results given in [HC76], [O77], and [Z78], show that among all possible “and”-operations $a, b \rightarrow \min(a, b)$ and $a, b \rightarrow ab$ are the best fit for human reasoning. The $\min$ operation does not seem to be adequate for our purposes, because if we use $\min$, then, e.g., the degree, to which a function $x(t)$ satisfies the condition “most likely, $|x(t)| \leq M$”, is equal to the minimal of the degrees of the corresponding statements. This minimum is attained when the value of $|x(t)|$ is the biggest possible. Therefore, the function $x_1(t)$ that is everywhere equal to $2M$, gets the same
degree of consistency with the above-given rule, as the function \( x_2(t) \) that is almost everywhere equal to 0, and is attaining the value \( 2M \) only on a small interval. Intuitively, however, for the first function \( x_1(t) \) (for which the inequality is always false), our degree of belief that it satisfies this condition is practically 0, while for the second function \( x_2(t) \), for which this inequality is almost everywhere true, our degree of belief must be close to 1. So using \( \min \) in our problem is inconsistent with our intuition, and therefore we must use the product for &.

Comment. Other arguments for choosing different & operations are given in [K83], [KR86], [K87], [KKM88], [K89], [K89a], [K90], [KK90], [KL90], [KQL91], [KQLFLKBR92].

What membership functions to choose? We want to describe the statements of the type “most likely, \(|X - a| \leq \delta\)”, where \(X\) is an unknown \((x(t), \hat{x}(t), \text{or } y(t))\) and \(a, \delta\) are known values (for example, \(\delta = M\) and \(a = 0\)). So we must describe, to what extent any given value \(x\) satisfies this condition.

Evidently, \(x\) satisfies the inequality \(|x - a| \leq \delta\) if and only if the value \(y = (x - a)/\delta\) satisfies the inequality \(|y| \leq 1\). Therefore, it is natural to assume that the statement “most likely, \(|X - a| \leq \delta\)” has the same degree of belief as the statement “most likely, \(|y| \leq 1\)”, where \(y = (x - a)/\delta\). So, if we will be able to describe a membership function \(\mu(y)\) that corresponds to the statement “most likely, \(|y| \leq 1\)”, then we will be able to describe our degree of belief \(\mu_1(x)\) that \(x\) satisfies the condition “most likely, \(|X - a| \leq \delta\)” as \(\mu((x - a)/\delta)\). So the main problem is to find an appropriate function \(\mu(x)\).

In the present paper we use Gaussian membership functions \(\mu(x) = \exp(-\beta x^2)\) for some \(\beta > 0\). Therefore, the statement “most likely, \(|X - a| \leq \delta\)” will be described by a membership function \(\mu_1(x) = \exp(-\beta(x - a)^2/\delta^2)\).

Gaussian membership functions are widely used in fuzzy systems and fuzzy control (see, e.g., [K75], [BCDMM85], [YIS85], [KM87, Ch. 5], etc.), and there are several theoretical explanations why they are so successful: in [KR86] and in Section 8 of [KQLFLKBR92] we prove that Gaussian functions are optimal (in some reasonable sense), and in [KQR92] we describe reasonable axioms that uniquely determine Gaussian membership functions.

A remark about defuzzification. Suppose that we have determined the membership functions \(\mu_i(x(t))\), that correspond to different statements about the unknown process \(x(t)\). Then the resulting membership function \(\mu(x(t))\) is obtained by multiplying the functions \(\mu_i(x(t))\) that correspond to these statements.

All the values of \(\mu_i\) are \(\leq 1\). So, if we multiply many such values, we end up with very small numbers. E.g., if we have 10 experts who all assign the truth value 0.9 to some event, the resulting estimate is \(0.9^{10} \approx 0.3\). Thus, the fact that for some process \(x(t)\) the membership value \(\mu(x(t))\) is small, does not necessarily mean that this particular dependency \(x(t)\) is hardly possible. What is meaningful is not the absolute, but the relative value of \(\mu(x(t))\): if \(\mu(x(t)) \ll \mu(y(t))\), then it does mean that, according or our knowledge, \(x(t)\) is much less probable than \(y(t)\).

To make these comparisons easier, L. Zadeh proposed to use normalization, i.e., turn from \(\mu(x(t))\) to \(\mu'(x(t)) = N\mu(x(t))\), where a normalizer \(N\) is chosen in such a way that the maximal value of \(\mu'(x(t))\) is equal to 1 (i.e., \(N = 1/(\max \mu(x(t)))\)).

Comment. Theoretical explanations of this choice of a normalization are given in [KQLFLKBR92] (in the framework of a general mathematical foundation scheme for fuzzy logic).
3. FUZZY DESCRIPTION OF RELEVANT EXPERTS KNOWLEDGE AND RESULTING REGULARIZATION

Let’s first list the possible experts’ statements.

1) Usually experts can give the approximate range of the process \( x(t) \), i.e., they can give a number \( M \) for which “most likely, for every \( t \) the value of \( |x(t)| \) is limited by \( M \).”

2) Usually they can also give some approximate bounds for the rate, with which the values of \( x(t) \) can change, i.e., they can give a number \( \Delta \), for which “most likely, for every \( t \), the value of \( |\dot{x}(t)| \) is limited by \( \Delta \).”

3) Sometimes, the experts’ knowledge and/or intuition can also prompt the approximate bounds for the second time derivative of the process (acceleration), and bounds for some higher derivatives. For each of these derivatives, an expert gives a value \( \Delta_i \) and states that “most likely, for every \( t \), the value of \( |x^{(i)}(t)| \) is limited by \( \Delta_i \)” (here \( x^{(i)}(t) \) denoted \( i \)-th derivative).

4) Experts can also give some information about the possible measurement errors, i.e., about the values \( n(t) = y(t) - \int k(t, s)x(s) \, ds \), where \( y(t) \) are the measured values. In this case, an expert gives a value \( \delta \), and states that “most likely, for every \( t \), the value of \( |n(t)| \) is limited by \( \delta \).”

In addition to that, we have some measurement results \( y(t) \), and these measurement results determine a set \( X \) of all the functions that are consistent with them. For example, if we know the maximal possible value \( \varepsilon \) of a measurement error \( n(t) \), then \( X \) consists of all the functions \( x(t) \) that satisfy the inequality \(|y(t) - \int k(t, s)x(s) \, ds| \leq \varepsilon \) for all \( t \).

We want to represent the expert knowledge in terms of a membership function that is defined on this set \( X \).

We cannot directly translate these statements into membership functions, so we need an additional approximation process. Each of these statements refers not to a single value of some variable, but to infinitely many values, namely, to the values of \( x(t) \) for all possible moments of time \( t \). So, if we write down all the resulting elementary statements, we will end up with infinitely many such statements. So, to get a membership function that corresponds to the resulting knowledge, we must apply an “and”-operator to infinitely many membership functions, that correspond to infinitely many elementary statements. But we know only how to apply “and”-operator to finitely many functions.

In order to cover the infinite case, we will apply the usual mathematical method of dealing with infinities: we will first consider the case, when the experts statements are applicable only to finitely many points \( t_1, \ldots, t_n \), and then tend \( n \) to infinity in such a way that in the limit these points \( t_i \) are everywhere dense. One of the natural possibilities to do that is to choose \( t_i = t_0 + ih \), where \( h > 0 \), and then take \( t_0 \to -\infty \), \( h \to 0 \), and \( n \to \infty \) in such a way that \( t_n = t_0 + nh \to +\infty \).

**The resulting membership function: derivation.** Let us apply this procedure and compute the resulting membership function. The readers who are interested only in the final result can skip this subsection.

Let’s first consider the case, when the only experts knowledge consists of the bounds \( M \) and \( \Delta \) on \( |x(t)| \) and \( |\dot{x}(t)| \). Then for each \( t \) the corresponding membership functions are \( \exp(-\beta |x(t)|^2/M^2) \) and \( \exp(-\beta |\dot{x}(t)|^2/\Delta^2) \). Therefore, if we take into consideration these statements for \( t = t_1, \ldots, t_n \), \( t_i = t_0 + ih \), the resulting membership function will be equal to the product of these membership functions, i.e., will be equal to the following expression

\[ \prod_{i=1}^{n} \left( \exp(-\beta |x(t)|^2/M^2) \right) \]
\[
\mu(x(t)) = \prod_{i=1}^{n} \exp(-\beta|x(t_i)|^2/M^2) \times \prod_{i=1}^{n} \exp(-\beta|x(t_i)|^2/\Delta^2)
\]

Comment. We are restricted to the set \(X\) of all functions \(x(t)\) that are consistent with the measurement results. Therefore, the above expression for \(\mu(x(t))\) is valid only for such functions \(x(t)\). All functions \(x(t)\) that are not consistent with the measurement results are impossible, i.e., if \(x(t) \notin X\), then \(\mu(x(t)) = 0\).

Since \(\exp(a) \times \exp(b) = \exp(a + b)\), we can simplify the expression for \(\mu(x(t))\) as follows:

\[
\mu(x(t)) = \exp(-\beta/M^2 \sum_{i=1}^{n} |x(t_i)|^2 - \beta/\Delta^2 \sum_{i=1}^{n} |\dot{x}(t_i)|^2).
\]

What happens when \(n \to \infty\)? If we multiply the sum \(\sum_{i=1}^{n} |x(t_i)|^2\) by \(h = t_{i+1} - t_i\), we get an integral sum for the integral \(\int |x(t)|^2 dt\). These integral sums tend to this integral, when \(h \to 0\). Hence, for small \(h\), this sum is approximately equal to \(h^{-1} \int |x(t)|^2 dt\). Therefore, the membership function is approximately equal to the following expression:

\[
\mu_h(x(t)) \approx \exp(-\beta/h)J(x(t)),
\]

where

\[
J(x(t)) = M^{-2} \int |x(t)|^2 dt + \Delta^{-2} \int |\dot{x}(t)|^2 dt.
\]

When \(h \to 0\), \((\beta/h)J(x(t)) \to \infty\), and, therefore, \(\mu_h(x(t)) \approx \exp(-\beta/h)J(x(t))\) \(\to 0\). Therefore, if we apply a transition to a limit, we end up with a meaningless expression \(\mu(x(t)) \equiv 0\).

In order to get a reasonable limit membership function \(\mu(x(t))\), we must apply the normalization procedure before going to a limit. In other words, we must transform \(\mu_h(x(t))\) into \(\mu'_h(x(t)) = N \mu_h(x(t))\), where \(N = 1/\max_{x(t) \in X} \mu_h(x(t))\).

Since \(\mu_h(x(t)) \approx \exp(-\beta/h)J(x(t))\), the value of \(\mu_h(x(t))\) is the biggest when the value of \(J(x(t))\) is the smallest possible. So, if we denote by \(m\) the smallest possible value of the functional \(J(x(t))\) on \(X\), we can conclude that \(\max_{x(t) \in X} \mu_h(x(t)) = \exp(-\beta/h)m\). Therefore, \(N = 1/\max = \exp((\beta/h)m)\), and \(\mu'_h(x(t)) = N \mu_h(x(t)) = \exp(-\beta/h)(J(x(t)) - m)\).

Now we are ready to describe the membership function \(\mu(x(t)) = \lim_{h \to 0} \mu'_h(x(t))\) that corresponds to the limit \(h \to 0\). If \(J(x(t)) = m\), then \(\mu'_h(x(t)) = 1\), and therefore \(\mu(x(t)) = 1\). If \(x(t) \in X\) and \(J(x(t)) \neq m\), then, since \(m\) is a minimum of \(J(x(t))\), we get \(J(x(t)) > m\), therefore \((\beta/h)(J(x(t)) - m) \to \infty\), and hence, \(\mu'_h(x(t)) \to 0\) as \(h \to 0\).

As a result, we get a crisp membership function that corresponds to Tikhonov’s regularization. If \(J(x(t)) \neq m\), we have \(\mu(x(t)) = 0\). So, although we started with fuzzy statements and fuzzy membership functions, the resulting membership function is crisp: it is equal either to 1 or to 0 depending on whether the functional \(J(x(t))\) attains its minimum at \(x(t)\) or not. Hence, in this case, we do not need any defuzzification procedure: we just pick a function \(x(t)\) from \(X\), for which \(J(x(t))\) attains its minimal value.

What if the experts can also give some bounds on the second and higher derivatives of the process \(x(t)\). In case an expert gives estimates \(\Delta_i\) for \(i\)-th derivative and/or a bound \(\delta\) for the measurement error, the resulting membership function is the same, with the only difference that additional terms are added to \(J(x(t))\): \(\Delta_i^{-2} \int (x^{(i)})^2 dt\) in case of \(i\)-th derivative, and \(\delta^{-2} \int (y(t) - \int k(t,s)x(s) ds)^2 dt\) in case of an error bound.
How to solve inverse problems: resulting procedure. As a result, we arrive at the following methods of solving inverse problems:

1) ask an expert to give approximate bounds \( M \) for \( |x(t)| \) and \( \Delta \) for \( |\dot{x}(t)| \); if possible, get also his bounds \( \Delta_i \) for \( i \)-th derivative \( |x^{(i)}(t)| \), and \( \delta \) for the measurement error \( |y(t) - \int k(t, s)x(s)\,ds| \);

2) from all the functions that are consistent with the measurement results, choose a function \( x(t) \) for which the functional \( J(x(t)) \) attains the smallest possible value. In case the expert gives only the estimates \( M \) and \( \Delta \), \( J(x(t)) = J_0(x(t)) + J_1(x(t)) \), where \( J_0(x(t)) = M^{-2} \int |x(t)|^2\,dt \) and \( J_1(x(t)) = \Delta^{-2} \int |\dot{x}(t)|^2\,dt \). In case he gives bounds for \( i \)-th derivative and/or for errors, we must take \( J(x(t)) = \sum_i J_i(x(t)) + J_e(x(t)) \), where for \( i > 1 \) \( J_i(x(t)) = \Delta_i^{-2} \int (x^{(i)}(t))^2\,dt \) and \( J_e(x(t)) = \delta^{-2} \int (y(t) - \int k(t, s)x(s)\,ds)^2\,dt \).

We can use ready-made software. The resulting method turns out to be a particular case of the Tikhonov’s regularization scheme. Therefore we do not need to design any new software: we can use the techniques, algorithms, and programs, that have already been developed for Tikhonov’s regularization.

If the only thing we have done is justification of a well-known method, then what’s the buzz? Our proposal to use Tikhonov’s method has two advantages over the usual heuristic suggestion to use it:

i) Tikhonov’s method is semi-heuristic, while we derived our method from the fuzzy formalism;

ii) we do not need any heuristic rule of choosing \( a_i \), because we have explicit expressions for these parameters in terms of experts’ bounds.

Therefore, we avoid the problem of Tikhonov’s regularization that different heuristic rules lead to different values of \( a_i \) and, therefore, to different solutions \( x(t) \).

4. CONCLUSIONS

Suppose that we must reconstruct \( x(t) \) from the measurement results \( y(t) \), and the problem is ill-posed in the sense that drastically different functions \( x(t) \) are consistent with the same measurement results. Such problems are very frequent in geophysics, astronomy, image processing, etc. Suppose also that the only additional information that we have about the process \( x(t) \) is the experts estimates \( M \) and \( \Delta \) for which the experts say that “most likely, for every \( t \) the value of \(|x(t)|\) is limited by \( M \),” and “most likely, for every \( t \), the value of \(|\dot{x}(t)|\) is limited by \( \Delta \),” where \( \dot{x}(t) \) denotes the rate with which \( x(t) \) changes (i.e., in mathematical terms, time derivative of \( x(t) \)).

Then fuzzy representation of this uncertainty leads to the following method of using this experts’ knowledge: from all the functions that are consistent with the measurement results, we choose a function \( x(t) \), for which the functional \( J(x(t)) \) takes the minimal possible value, where \( J(x(t)) = M^{-2} \int |x(t)|^2\,dt + \Delta^{-2} \int |\dot{x}(t)|^2\,dt \).

Similar functionals can be described for the cases, when bounds for higher derivatives and/or measurement errors are known.

The resulting method turns out to coincide with a particular case of the general Tikhonov’s regularization approach. This approach has already been implemented in software, and it has been successfully tested on numerous real-life ill-posed problems.
The advantage of our approach is that we solve two main problems of Tikhonov's regularization:

- we provide a justification its formulas, and
- we provide a method for choosing the parameters of Tikhonov's regularization.

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